Addendum

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Proof of Remark 4.11.(ii) in
Surface Diagrams for
Frobenius Algebras and Frobenius-Schur Indicators
in Grothendieck-Verdier Categories

Let $\mathcal{C} = (\mathcal{C}, \otimes, 1, K)$ and $\mathcal{D} = (\mathcal{D}, \otimes, 1, K)$ be GV-categories. As stated in [DS25, Prop. 2.50], their duality functors are strong Frobenius LD-functors $\mathcal{C} \to \mathcal{C}^{\mathrm{lop}}$ and $\mathcal{D} \to \mathcal{D}^{\mathrm{lop}}$. For the definition of Frobenius LD-functors, refer to [DS25, Def. 2.13]. From now on, by abuse of notation, we denote both duality functors by D.

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a Frobenius LD-functor. We consider its multiplication morphism

$$\varphi^2\colon \otimes \circ (F \times F) \to F \circ \otimes,$$

unit morphism $\varphi^0\colon 1 \to F(1)$, comultiplication morphism $\nu^2\colon F \circ ⅋ \to ⅋ \circ (F \times F)$, and counit morphism $\nu^0\colon F(K) \to K$.

Proposition 0.1. *The duality transformation of F*

$$\xi^F\colon\ F \circ D \xrightarrow{\simeq} D \circ F,$$

as defined in [DS25, Def. 4.10], is a morphism of Frobenius LD-functors.

Proof. We will prove that the duality transformation ξ^F is an $\otimes$-monoidal natural transformation. The proof that ξ^F is also ⅋-monoidal follows by applying the 2-functor $(-)^{\mathrm{cop}}$, as defined in [DS25, Def. 2.24], to the surface diagrams used in the $\otimes$-monoidal proof.

As a preliminary remark, observe that, as in the rigid case, two morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, D(B))$ are equal if and only if we have

$$\mathrm{ev}_B \circ (f \otimes B) \ = \ \mathrm{ev}_B \circ (g \otimes B). \tag{0.1}$$

This fact can be proven surface-diagrammatically using the snake equation (S1). We will utilize this fact in the remainder of the proof.

First, we show that the following diagram commutes for all objects $X, Y \in \mathcal{C}$:

$$\begin{tikzcd}
FD(X) \otimes FD(Y) \arrow[r, "\varphi^2_{DX,DY}"] \arrow[d, "\xi^F_X \otimes \xi^F_X"'] & F(DX \otimes DY) \arrow[r, "\simeq"] & FD(Y ⅋ X) \arrow[d, "\xi^F_{Y ⅋ X}"] \\
DF(X) \otimes DF(Y) \arrow[r, "\simeq"'] & D(F(Y) ⅋ F(X)) \arrow[r, "D(\nu^2_{Y,X})"'] & DF(Y ⅋ X).
\end{tikzcd} \tag{0.2}$$

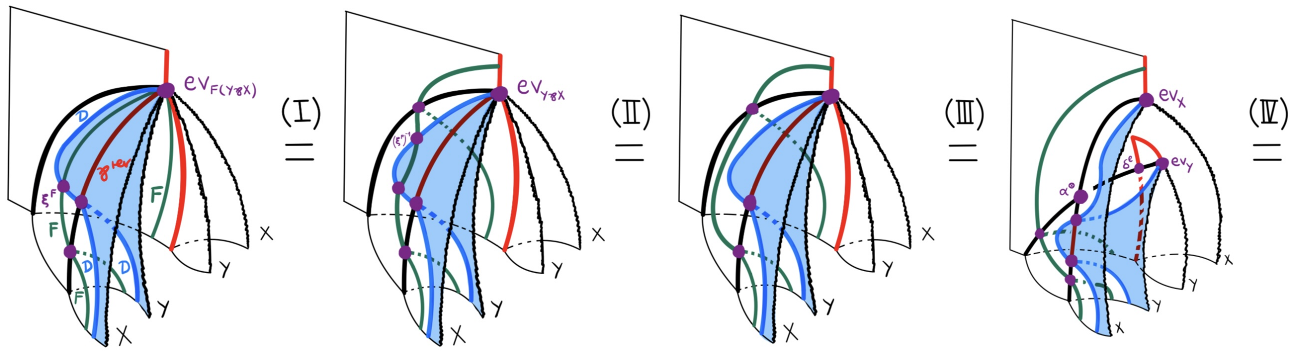
Here, the unlabeled isomorphisms are structure morphisms of the Frobenius LD-functor D.

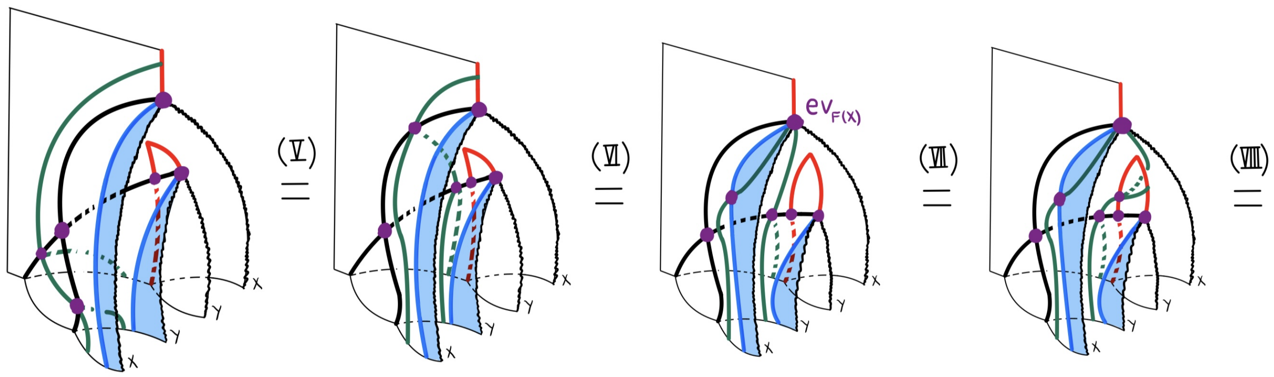
To prove commutativity, we use surface diagrams. In the following computation, the first surface diagram represents the morphism

$$\big(FD(X) \otimes FD(Y)\big) \otimes F(Y \parr X) \longrightarrow K,$$

which is built from the clockwise morphism in diagram (0.2), and the evaluation

$$\mathrm{ev}_{F(Y \parr X)} \colon\ DF(Y \parr Y) \otimes F(Y \parr X) \overset{\mathsf{def}}{=} DF(X \parr^{\mathrm{rev}} Y) \otimes F(Y \parr X) \longrightarrow K.$$





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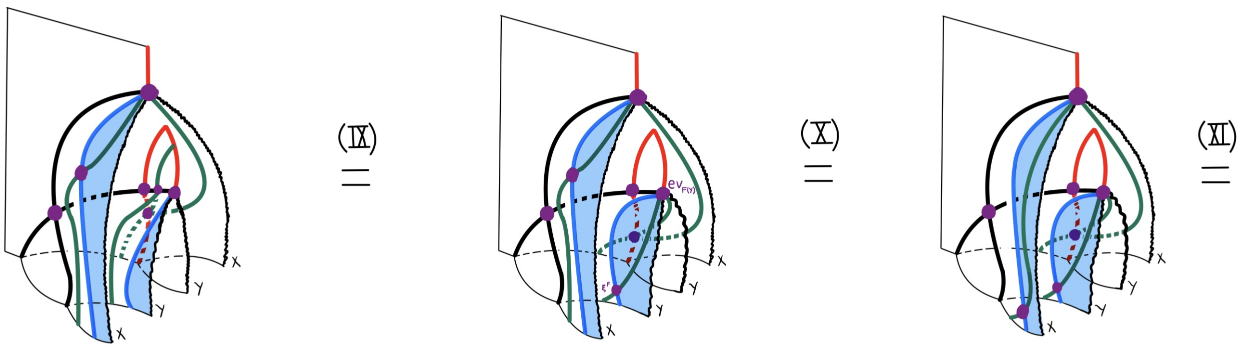

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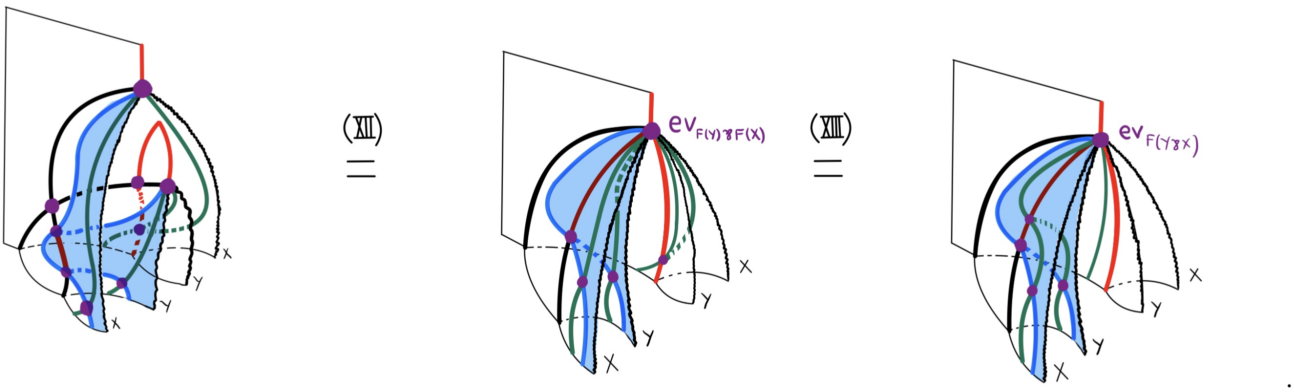


Equations (I), (VI) and (IX) hold by the defining equation of the inverse duality transformation $(\xi^F)^{-1}$; see [DS25, Remark 4.11.(i)]. Equation (II) follows from the invertibility of the duality transformation. For Equation (III), consider the composite

$$D(Y \parr X) \otimes (Y \parr X) \;\cong\; \big(DX \otimes DY\big) \otimes (Y \parr X) \xrightarrow{\overline{\mathrm{ev}}} K, \tag{0.3}$$

where we define

$$\overline{\mathrm{ev}} \;:=\; \mathrm{ev}_X \circ \big(DX \otimes l^{\parr}_X\big) \circ \big(DX \otimes (\mathrm{ev}_Y \parr X)\big) \circ (DX \otimes \delta^l_{DY,Y,X}) \circ \alpha^{\otimes}_{DX,DY,Y \parr X}.$$

One can show that this composite is a side inverse, in the sense of [DS25, Def. 2.26], to the composite

$$1 \xrightarrow{\overline{\mathrm{coev}}} (Y \parr X) \parr (DX \otimes DY) \;\cong\; (Y \parr X) \parr D(Y \parr X),$$

where we set

$$\overline{\mathrm{coev}} \;:=\; (\alpha^{\parr}_{Y,X,DX\otimes DY})^{-1} \circ (Y \parr \delta^r_{X,DX,DY}) \circ \big(Y \parr (\mathrm{coev}_X \otimes DY)\big) \circ \big(Y \parr (l^{\otimes}_{DY})^{-1}\big) \circ \mathrm{coev}_Y.$$

Thus, we define the evaluation $\mathrm{ev}_{Y \parr X} \colon D(Y \parr X) \otimes (Y \parr X) \to K$ as the composite in Equation (0.3). With this choice, Equation (III) holds. Equations (IV) and (XI) follow since the Frobenius LD-structure on the duality functor D is strong. Equation (V) is an instance of the associativity of the lax $\otimes$-monoidal functor F. Equation (VII) follows from the counitality axiom of the $\parr$-monoidal functor F. Equation (VIII) holds by the Frobenius relation (F1) of the Frobenius LD-functor F. Equation (X) amounts to pushing down the duality transformation ξ^F, and is valid by the axioms of a strict monoidal 2-category. Equation (XII) follows analogously to Equation (III). Finally, Equation (XIII) holds by [DS25, Prop. 2.46].

By Equation (0.1), the surface-diagrammatic computation shows that diagram (0.2) commutes.

Next, we show that the following diagram commutes:

$$\begin{tikzcd} 1 \arrow[r, "\varphi^0"] \arrow[d, "\simeq"'] & F(1) \arrow[r, "\simeq"] & FD(K) \arrow[d, "\xi^F_K"] \\ D(K) \arrow[rr, "D(\nu^0)"'] & & DF(K). \end{tikzcd} \tag{0.4}$$

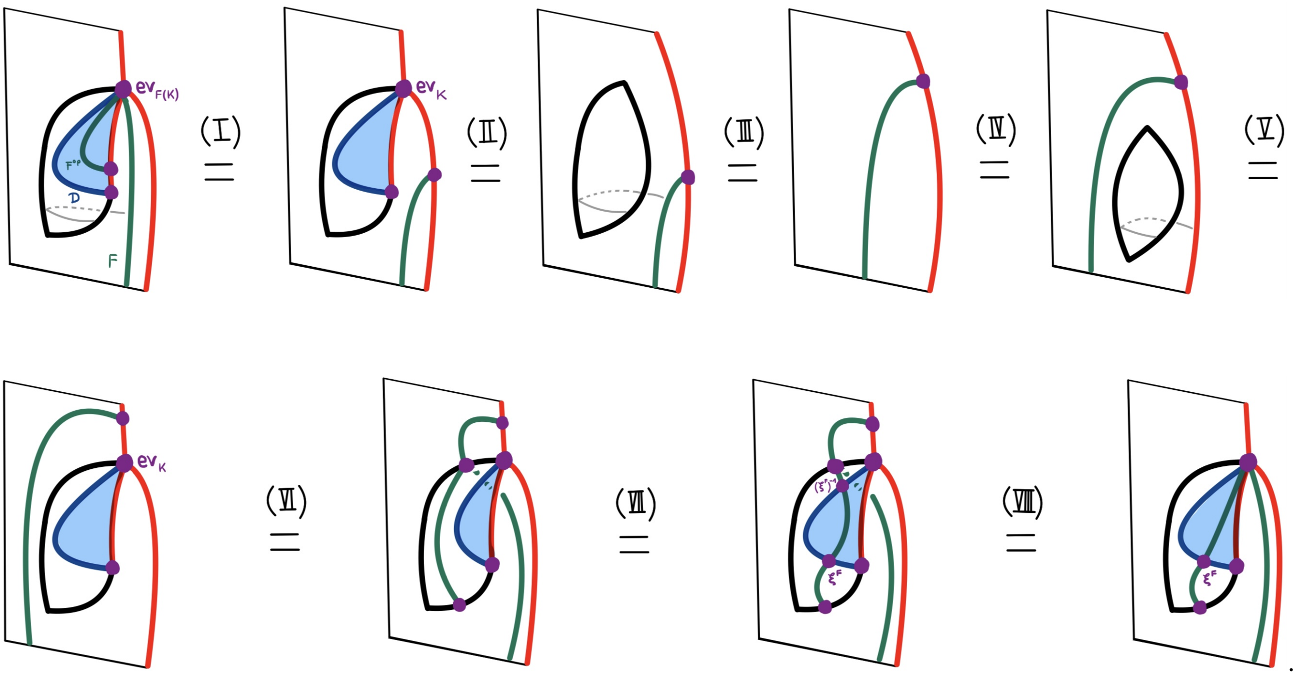
Again, the unlabelled isomorphisms are structure morphisms of the Frobenius LD-functor D.

To prove commutativity we use surface diagrams. In the following computation, the first surface diagram represents the morphism

$$F(K) \xrightarrow{\simeq} 1 \otimes F(K) \longrightarrow DF(K) \otimes F(K) \longrightarrow K,$$

which is built from the inverse left unitor, the counterclockwise morphism in diagram (0.2), and the evaluation

$$\mathrm{ev}_{F(K)} \colon\ F(K) \otimes K \longrightarrow K.$$



Equation (I) holds by [DS25, Prop. 2.46]. By [DS25, Ex. 2.28], we can assume that we have $D(1) = K$, $D(K) = 1$, and $\mathrm{ev}_K = l_K^{\otimes}$. With this choice, Equations (II) and (V) hold. Equation (III) and (IV) follow from the invertibility of the unitors. The unitality of the lax $\otimes$-monoidal functor F implies Equation (VI). Equation (VII) holds by the invertibility of the duality transformation ξ^F, while Equation (VIII) follows from the defining equation of the inverse duality transformation $(\xi^F)^{-1}$; see [DS25, Remark 4.11.(i)].

Finally, by Equation (0.1) and the invertibility of the left unitor, the above calculation implies that diagram (0.4) commutes.

This completes the proof that the duality transformation ξ^F is an $\otimes$-monoidal natural transformation. $\square$

References

[DS25] M. Demirdilek and C. Schweigert. Surface Diagrams for Frobenius Algebras and Frobenius-Schur Indicators in Grothendieck-Verdier Categories, 2025. Preprint. arXiv:2503.13325.