

Bases in standard categories

Notes based on the paper

Cellularity of endomorphism algebras of tilting objects

by Gwyn Bellamy and Ulrich Thiel

Max Demirdilek

January 2023

Contents

1	Introduction	2
2	Tools: Lifts and extensions	3
3	Basic construction	5
4	Basis theorem(s)	6
5	Combinatorial bases	12
5.1	Standard bases	12
5.2	Cellular bases	14

1 Introduction

In this talk, we construct certain combinatorial bases of the endomorphism algebra of a tilting object in a standard category over a field. The talk closely follows [BT22, Section 3]. Consequently, for the most part we employ the notational conventions of [BT22]. The quoted article [BT22] generalises the construction of [AST18] from the context of quantum enveloping algebras to a broader categorical framework. More precisely, the category of modules over a quantum enveloping algebra is replaced by an arbitrary so-called standard category over a field.

For completeness, we recall the definition of a standard category over a field.

Definition 1.1

Let K be a field. Let Λ be a partially ordered set. A category \mathcal{C} is called a standard category over K with respect to Λ if it satisfies the following four assumptions:

1. (Basic properties) The category \mathcal{C} is an essentially small, locally finite, K -linear, abelian category. By locally finite we mean on the one hand that all objects in \mathcal{C} have finite length. On the other hand, the hom-space $\text{Hom}_{\mathcal{C}}(X, Y)$ is required to be a finite-dimensional K -vector space for all objects $X, Y \in \mathcal{C}$.
2. (Simple objects) There exists a set $\{L(\lambda)\}_{\lambda \in \Lambda}$ of simple objects in \mathcal{C} such that the function

$$\begin{aligned} \Lambda &\longrightarrow \{\text{isomorphism classes of simple objects in } \mathcal{C}\} \\ \lambda &\longmapsto \text{isomorphism class of } L(\lambda) \end{aligned}$$

is a bijection.

3. (Ext-vanishing condition) Each $L(\lambda)$ has a costandard object $\nabla(\lambda)$ and a standard object $\Delta(\lambda)$ such that the following Ext-vanishing condition holds for every $i \in \{0, 1, 2\}$ and all $\lambda, \mu \in \Lambda$:

$$\text{Ext}_{\mathcal{C}}^i(\Delta(\lambda), \nabla(\mu)) \cong \begin{cases} K & \text{if } i = 0 \text{ and } \lambda = \mu \\ 0 & \text{else} \end{cases}$$

4. (Indecomposable tilting objects)

- (a) For any $\lambda \in \Lambda$, there is an indecomposable tilting object $T(\lambda)$ such that:
 - i. $T(\lambda)$ is of highest weight λ ,
 - ii. There is a monomorphism $\Delta(\lambda) \hookrightarrow T(\lambda)$,
 - iii. There is an epimorphism $T(\lambda) \twoheadrightarrow \nabla(\lambda)$.

- (b) The function

$$\begin{aligned} \Lambda &\longrightarrow \{\text{isomorphism classes of indecomposable tilting objects in } \mathcal{C}\} \\ \lambda &\longmapsto \text{isomorphism class of } T(\lambda) \end{aligned}$$

is a bijection.

Remark 1.2. (Terminology)

We will mostly call a ‘standard category over K with respect to Λ ’ simply a ‘standard category over K ’.

Recall Proposition 3.2.5 from the previous talk by Kian.

Proposition 1.3. (Donkin's Ext-criteria)

Let \mathcal{C} be a standard category over K . Then the following two equivalences hold:

- (i) $M \in \mathcal{C}^\Delta$ if and only if $\text{Ext}_{\mathcal{C}}^1(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$.
- (ii) $N \in \mathcal{C}^\nabla$ if and only if $\text{Ext}_{\mathcal{C}}^1(\Delta(\lambda), N) = 0$ for all $\lambda \in \Lambda$.

Corollary 1.4.

Let \mathcal{C} be a standard category over K . Let $M \in \mathcal{C}^\Delta$ and $N \in \mathcal{C}^\nabla$. Then $\text{Ext}_{\mathcal{C}}^1(M, N) = 0$.

Proof.

We proceed by induction on the length of a costandard filtration of N . The base case follows immediately from Proposition 1.3. For the induction step, let

$$0 = N_0 \subsetneq \cdots \subsetneq N_{n-1} \subsetneq N_n = N \quad (1)$$

be a costandard filtration of $N \in \mathcal{C}$. Let $\lambda \in \Lambda$ such that $N_n/N_{n-1} \cong \nabla(\lambda)$. Applying the functor $\text{Hom}_{\mathcal{C}}(M, -)$ to the short exact sequence

$$0 \rightarrow N_{n-1} \rightarrow N \rightarrow \nabla(\lambda) \rightarrow 0 \quad (2)$$

yields an exact sequence

$$0 \rightarrow \cdots \rightarrow \text{Ext}_{\mathcal{C}}^1(M, N_{n-1}) \rightarrow \text{Ext}_{\mathcal{C}}^1(M, N) \rightarrow \text{Ext}_{\mathcal{C}}^1(M, \nabla(\lambda)) \rightarrow \cdots. \quad (3)$$

By the induction hypothesis we know that $\text{Ext}_{\mathcal{C}}^1(M, N_{n-1}) = 0$ holds. Moreover, by Proposition 1.3 we have $\text{Ext}_{\mathcal{C}}^1(M, \nabla(\lambda)) = 0$. Since the sequence 3 is exact, it follows that $\text{Ext}_{\mathcal{C}}^1(M, N) = 0$. By induction the claim follows. \square

2 Tools: Lifts and extensions

For the remainder of this talk, we fix a field K and a standard category \mathcal{C} over K . Additionally, we fix objects $M \in \mathcal{C}^\Delta$ and $N \in \mathcal{C}^\nabla$.

For every $\lambda \in \Lambda$, we choose an epimorphism $\pi^\lambda: T(\lambda) \twoheadrightarrow \nabla(\lambda)$ and a monomorphism $\iota^\lambda: \Delta(\lambda) \hookrightarrow T(\lambda)$. These exist by Definition 1.1.4. For every $\lambda \in \Lambda$, define $c^\lambda := \pi^\lambda \circ \iota^\lambda$.

Informally, we will construct bases of $\text{Hom}_{\mathcal{C}}(M, N)$ by 'factorizing through the indecomposable tilting objects $\{T(\lambda)\}_{\lambda \in \Lambda}$ of \mathcal{C} '. Each factoring morphism will be realized either as a lift along π^λ or as an extension along ι^λ .

The following lemma will be the main tool in the subsequent construction of bases of $\text{Hom}_{\mathcal{C}}(M, N)$.

Lemma 2.1. (Lifting lemma)

Let $\lambda \in \Lambda$.

- (i) Any morphism $f: M \rightarrow \nabla(\lambda)$ lifts to a morphism $\tilde{f}: M \rightarrow T(\lambda)$ along $\pi^\lambda: T(\lambda) \twoheadrightarrow \nabla(\lambda)$.
As a commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\quad \exists \tilde{f} \quad} & T(\lambda) \\ & \searrow f & \downarrow \pi^\lambda \\ & & \nabla(\lambda) \end{array}$$

(ii) Dually, any morphism $g: \Delta(\lambda) \rightarrow N$ extends to a morphism $\tilde{g}: T(\lambda) \rightarrow N$ along $\iota^\lambda: \Delta(\lambda) \hookrightarrow T(\lambda)$. As a commutative diagram:

$$\begin{array}{ccc}
 \Delta(\lambda) & & \\
 \downarrow \iota^\lambda & \searrow g & \\
 T(\lambda) & \xrightarrow{\exists \tilde{g}} & N
 \end{array}$$

Proof.

We only prove the first part of the lemma; the second one is shown dually.

Consider the short exact sequence

$$0 \rightarrow \ker(\pi^\lambda) \rightarrow T(\lambda) \xrightarrow{\pi^\lambda} \nabla(\lambda) \rightarrow 0. \quad (4)$$

We claim that the object $\ker(\pi^\lambda) \in \mathcal{C}$ admits a costandard filtration. By Donkin's Ext-criteria 1.3 this amounts to proving that $\text{Ext}_{\mathcal{C}}^1(\Delta(\mu), \ker(\pi^\lambda)) = 0$ for all $\mu \in \Lambda$. The subsequent proof of this claim was suggested by Paul Wedrich. Let $\mu \in \Lambda$. Applying the functor $\text{Hom}_{\mathcal{C}}(\Delta(\mu), -)$ to the short exact sequence 4 yields an exact sequence

$$\text{Hom}_{\mathcal{C}}(\Delta(\mu), T(\lambda)) \xrightarrow{\pi_*^\lambda} \text{Hom}_{\mathcal{C}}(\Delta(\mu), \nabla(\lambda)) \rightarrow \text{Ext}_{\mathcal{C}}^1(\Delta(\mu), \ker(\pi^\lambda)) \rightarrow \text{Ext}_{\mathcal{C}}^1(\Delta(\mu), T(\lambda)). \quad (5)$$

Note that by Donkin's Ext-criteria 1.3, we have $\text{Ext}_{\mathcal{C}}^1(\Delta(\mu), T(\lambda)) = 0$. We make a case distinction. If $\mu \neq \lambda$, then by Definition 1.1.3, we have $\text{Hom}_{\mathcal{C}}(\Delta(\mu), \nabla(\lambda)) = 0$. Since sequence 5 is exact, we conclude $\text{Ext}_{\mathcal{C}}^1(\Delta(\mu), \ker(\pi^\lambda)) = 0$. If on the other hand $\mu = \lambda$, then by Proposition 3.2.7.3 from Kian's talk, the morphism

$$\pi_*^\lambda: \text{Hom}_{\mathcal{C}}(\Delta(\lambda), T(\lambda)) \rightarrow \text{Hom}_{\mathcal{C}}(\Delta(\lambda), \nabla(\lambda)) \quad (6)$$

is surjective. By the exactness of sequence 5 and since $\text{Ext}_{\mathcal{C}}^1(\Delta(\lambda), T(\lambda)) = 0$ holds, this implies $\text{Ext}_{\mathcal{C}}^1(\Delta(\lambda), \ker(\pi^\lambda)) = 0$. Thus, the object $\ker(\pi^\lambda) \in \mathcal{C}$ admits a costandard filtration.

Since by assumption, $M \in \mathcal{C}$ admits a standard filtration, it follows from Corollary 1.4 that $\text{Ext}_{\mathcal{C}}^1(M, \ker(\pi^\lambda)) = 0$. Consequently, by applying the left exact functor $\text{Hom}_{\mathcal{C}}(M, -)$ to sequence 4 we obtain the exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(M, \ker(\pi^\lambda)) \rightarrow \text{Hom}_{\mathcal{C}}(M, T(\lambda)) \xrightarrow{\pi_*^\lambda} \text{Hom}_{\mathcal{C}}(M, \nabla(\lambda)) \rightarrow \text{Ext}_{\mathcal{C}}^1(M, \ker(\pi^\lambda)) = 0.$$

Thus, the map

$$\pi_*^\lambda: \text{Hom}_{\mathcal{C}}(\Delta(M, T(\lambda)) \rightarrow \text{Hom}_{\mathcal{C}}(M, \nabla(\lambda))$$

is surjective. □

Remark 2.2.

In general, the lifts and extensions of the lifting lemma 2.1 are not unique. In other words, post-composition with π^λ (pre-composition with ι^λ) is not injective in general. Moreover, the proof of the lifting lemma 2.1 gives a criterion for the injectivity of the map π_*^λ . Namely, the map π_*^λ is injective if and only if $\text{Hom}_{\mathcal{C}}(M, \ker(\pi^\lambda)) = 0$.

3 Basic construction

By employing the lifting lemma 2.1, we will now construct bases of the finite-dimensional K -vector space $\text{Hom}_{\mathcal{C}}(M, N)$.

Construction 3.1.

The construction consists of six steps.

(1) For every $\lambda \in \Lambda$, choose a basis F_M^λ of the finite-dimensional K -vector space $\text{Hom}_{\mathcal{C}}(M, \nabla(\lambda))$.

(2) Next, for every $\lambda \in \Lambda$ and every basis element $f \in F_M^\lambda$ choose a lift $\tilde{f} \in \text{Hom}_{\mathcal{C}}(M, T(\lambda))$ along π^λ . Such a lift exists by the lifting lemma 2.1. Denote the set of lifts by

$$\tilde{F}_M^\lambda := \{\tilde{f} \mid f \in F_M^\lambda\}.$$

(3) Dually, for every $\lambda \in \Lambda$, choose a basis G_N^λ of the finite-dimensional K -vector space $\text{Hom}_{\mathcal{C}}(\Delta(\lambda), N)$.

(4) For every $\lambda \in \Lambda$ and every basis element $g \in G_N^\lambda$ pick an extension $\tilde{g} \in \text{Hom}_{\mathcal{C}}(T(\lambda), N)$ along ι^λ . Denote the set of extensions by

$$\tilde{G}_N^\lambda := \{\tilde{g} \mid g \in G_N^\lambda\}.$$

(5) Next, for every $\lambda \in \Lambda$, define the subset

$$\tilde{G}_N^\lambda \tilde{F}_M^\lambda := \{\tilde{g} \circ \tilde{f} \mid \tilde{g} \in \tilde{G}_N^\lambda \text{ and } \tilde{f} \in \tilde{F}_M^\lambda\} \subseteq \text{Hom}_{\mathcal{C}}(M, N).$$

Here, we employ the following convention: If one of the sets \tilde{G}_N^λ or \tilde{F}_M^λ is empty, then we let the set $\tilde{G}_N^\lambda \tilde{F}_M^\lambda$ be empty.

One may illustrate the set $\tilde{G}_N^\lambda \tilde{F}_M^\lambda$ by the subsequent commutative diagram:

$$\begin{array}{ccccc}
& & \Delta(\lambda) & & \\
& \iota^\lambda \downarrow & & & \\
M & \xrightarrow{\tilde{f}} & T(\lambda) & \xrightarrow{\tilde{g}} & N \\
& f \searrow & \pi^\lambda \downarrow & & \\
& & \nabla(\lambda) & &
\end{array}$$

(6) Finally, define the following set

$$\tilde{G}_N \tilde{F}_M := \bigcup_{\lambda \in \Lambda} \tilde{G}_N^\lambda \tilde{F}_M^\lambda \subseteq \text{Hom}_{\mathcal{C}}(M, N).$$

4 Basis theorem(s)

The next theorem is the main result of this section.

Theorem 4.1. (Basis theorem)

The set $\tilde{G}_N \tilde{F}_M$ is a basis of the K -vector space $\text{Hom}_{\mathcal{C}}(M, N)$.

Before we prove the basis theorem, we first show a weaker version. To be able to formulate and prove the weaker version we need a few additional definitions and results.

Definition 4.2

Let $\phi \in \text{Hom}_{\mathcal{C}}(M, N)$ be a morphism. Let $\mu \in \Lambda$. We call $\phi_\mu := [\text{im}(\phi) : L(\mu)] \in \mathbb{N}$ the μ -multiplicity of ϕ .

Lemma 4.3. (Subadditivity of the μ -multiplicity)

Let $\phi, \psi \in \text{Hom}_{\mathcal{C}}(M, N)$ and $\mu \in \Lambda$. Then we have $(\phi + \psi)_\mu \leq \phi_\mu + \psi_\mu$.

The proof of Lemma 4.3 employs a corollary of the following proposition.

Proposition 4.4.

Let $\lambda \in \Lambda$. Let

$$0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0$$

be a short exact sequence in \mathcal{C} . Then the following equality holds

$$[Z : L(\lambda)] = [X : L(\lambda)] + [Y : L(\lambda)].$$

Proof.

The proposition immediately follows from the proof of [P70, Corollary 4.5.2].

More specifically, let

$$0 = Z_0 \subsetneq \cdots \subsetneq Z_i = X \subsetneq \cdots \subsetneq Z_n = Z$$

be a composition series of Z through X . Such a composition series exists by [P70, Proposition 4.5.1]. Note that for all $k \in \{i, \dots, n-1\}$ we have

$$(Z_{k+1}/X)/(Z_k/X) \cong Z_{k+1}/Z_k \tag{7}$$

by one of the isomorphism theorems. In particular, the object $(Z_{k+1}/X)/(Z_k/X)$ is simple for all $k \in \{i, \dots, n-1\}$. Thus, $0 = Z_i/X \subsetneq \cdots \subsetneq Z_n/X \cong Y$ is a composition series of Y . Now, equation 7 immediately yields $[Z : L(\lambda)] = [X : L(\lambda)] + [Y : L(\lambda)]$. □

Corollary 4.5.

Let $\lambda \in \Lambda$. Let $X, Y \in \mathcal{C}$ be objects. Then the following hold:

- (i) If there exists a monomorphism $X \hookrightarrow Y$, then we have $[Y : L(\lambda)] \geq [X : L(\lambda)]$.
- (ii) If there exists an epimorphism $Y \twoheadrightarrow X$, then we have $[Y : L(\lambda)] \geq [X : L(\lambda)]$.
- (iii) We have $[X \oplus Y : L(\lambda)] = [X : L(\lambda)] + [Y : L(\lambda)]$.

Proof of Lemma 4.3.

Let $\phi, \psi \in \text{Hom}_{\mathcal{C}}(M, N)$ and $\mu \in \Lambda$. Firstly, note that $\text{im}(\phi + \psi) \subseteq \text{im}(\phi) \cup \text{im}(\psi)$ by [P70, Lemma 4.3.6.]. Here, $\text{im}(\phi) \cup \text{im}(\psi)$ denotes the join in the preorder of subobjects of N . The join exists in any abelian category. Additionally, the two canonical morphisms

$$\begin{aligned} \text{im}(\phi) &\rightarrow \text{im}(\phi) \cup \text{im}(\psi) \\ \text{im}(\psi) &\rightarrow \text{im}(\phi) \cup \text{im}(\psi) \end{aligned}$$

induce an epimorphism

$$\text{im}(\phi) \oplus \text{im}(\psi) \rightarrow \text{im}(\phi) \cup \text{im}(\psi).$$

Thus, by Corollary 4.5 we have for all $\mu \in \Lambda$:

$$\begin{aligned} (\phi + \psi)_{\mu} &\stackrel{\text{def}}{=} [\text{im}(\phi + \psi) : L(\mu)] \leq [\text{im}(\phi) \cup \text{im}(\psi) : L(\mu)] \leq [\text{im}(\phi) \oplus \text{im}(\psi) : L(\mu)] \\ &= [\text{im}(\phi) : L(\mu)] + [\text{im}(\psi) : L(\mu)] \\ &\stackrel{\text{def}}{=} \phi_{\mu} + \psi_{\mu}. \end{aligned}$$

Next, given $\lambda \in \Lambda$, define two sets □

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda} &:= \{\phi \in \text{Hom}_{\mathcal{C}}(M, N) \mid \phi_{\mu} \neq 0 \text{ implies } \mu \leq \lambda\}, \\ \text{Hom}_{\mathcal{C}}(M, N)^{< \lambda} &:= \{\phi \in \text{Hom}_{\mathcal{C}}(M, N) \mid \phi_{\mu} \neq 0 \text{ implies } \mu < \lambda\}. \end{aligned}$$

Remark 4.6. (A filtration-like structure)

Let $\mu, \lambda \in \Lambda$. If $\mu \leq \lambda$ holds, then we have the subsequent inclusions:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(M, N)^{\leq \mu} &\subseteq \text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}, \\ \text{Hom}_{\mathcal{C}}(M, N)^{< \mu} &\subseteq \text{Hom}_{\mathcal{C}}(M, N)^{< \lambda}. \end{aligned}$$

Proposition 4.7.

Let $\lambda \in \Lambda$.

- (i) The set $\text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$ is a K -vector subspace of $\text{Hom}_{\mathcal{C}}(M, N)$.
- (ii) Similarly, the set $\text{Hom}_{\mathcal{C}}(M, N)^{< \lambda}$ is a K -vector subspace of $\text{Hom}_{\mathcal{C}}(M, N)$.

Proof.

We only show statement (i). Statement (ii) is proved analogously.

Clearly, we have $0_{MN} \in \text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$. Next, we convince ourselves that the set $\text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$ is closed under scalar multiplication. To do so, let $k \in K$ and $\text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$. Since the category \mathcal{C} is K -linear, we have

$$\text{im}(k \cdot \phi) \cong \begin{cases} \text{im}(\phi) & \text{if } k \neq 0 \\ 0 & \text{else} \end{cases} \quad (8)$$

Hence, $k \cdot \phi \in \text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$.

Next, we argue that $\text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$ is closed under addition. Let $\phi, \psi \in \text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$. Let $\mu \in \Lambda$ such that $\mu \not\leq \lambda$. By the subadditivity of the μ -multiplicity (i.e. Lemma 4.3), we then have:

$$0 \leq (\phi + \psi)_{\mu} \leq \phi_{\mu} + \psi_{\mu} = 0 + 0 = 0. \quad (9)$$

□

Lemma 4.8.

Let $\lambda \in \Lambda$. If $\phi, \psi \in \text{Hom}_{\mathcal{C}}(M, N)$ and $\phi_\lambda = 0$, then $(\phi + \psi)_\lambda = \psi_\lambda$.

Proof.

Consider the map $q := \text{coker}(\phi): N \rightarrow N/\text{im}(\phi)$. Set $\bar{\phi} = q \circ \phi$ and $\overline{\phi + \psi} = q \circ (\phi + \psi)$. Since in an abelian category composition is bilinear, we have $\overline{\phi + \psi} = \bar{\psi}$. Moreover, we claim that $(\phi + \psi)_\lambda = (\overline{\phi + \psi})_\lambda$ and $\psi_\lambda = \bar{\psi}_\lambda$. We only show $\psi_\lambda = \bar{\psi}_\lambda$. The other equality is proven analogously. By Corollary 4.5 we know that

$$0 \leq [\text{im}(\psi) \cap \text{im}(\phi): L(\lambda)] \leq [\text{im}(\phi): L(\lambda)] \stackrel{\text{def}}{=} \phi_\lambda = 0. \quad (10)$$

Consider the following short exact sequence

$$0 \rightarrow \text{im}(\phi) \cap \text{im}(\psi) \xrightarrow{\text{inc}} \text{im}(\psi) \xrightarrow{q_{|\text{im}(\psi)}} \text{im}(q_{|\text{im}(\psi)}) = \text{im}(q \circ \psi) \rightarrow 0.$$

Here, the morphism $q_{|\text{im}(\psi)}: \text{im}(\psi) \rightarrow N/\text{im}(\phi)$ is defined as the composition of morphisms $q_{|\text{im}(\psi)} := q \circ m$, where $m: \text{im}(\psi) \hookrightarrow N$ denotes the canonical monomorphism. With Proposition 4.4 and equation 10 we conclude $\psi_\lambda = \bar{\psi}_\lambda$.

In total, we find

$$(\phi + \psi)_\lambda = (\overline{\phi + \psi})_\lambda = \bar{\psi}_\lambda = \psi_\lambda.$$

□

We are now finally able to prove the weaker version of the basis theorem 4.1.

Proposition 4.9. (Dependent basis theorem)

For every $\mu \in \Lambda$, choose a basis F_M^μ and a corresponding set of lifts \tilde{F}_M^μ . Then for all $\mu \in \Lambda$, there exists a basis G_N^μ and a set of extensions \tilde{G}_N^μ satisfying the following properties:

- (i) The set $\tilde{G}_N \tilde{F}_M$ is a basis of the K -vector space $\text{Hom}_{\mathcal{C}}(M, N)$.
- (ii) Let $\mu \in \Lambda$. If ϕ is a non-zero morphism in the K -span $\text{span}_K(\tilde{G}_N^\mu \tilde{F}_M^\mu) \subseteq \text{Hom}_{\mathcal{C}}(M, N)$ of $\tilde{G}_N^\mu \tilde{F}_M^\mu$, then $\phi_\mu \neq 0$.

Additionally, any choice of G_N^μ and \tilde{G}_N^μ that satisfies properties (i) and (ii) also satisfies the following two properties:

- (iii) For all $\lambda \in \Lambda$, the set $\bigcup_{\mu \leq \lambda} \tilde{G}_N^\mu \tilde{F}_M^\mu$ is a K -vector space basis of $\text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$.
Similarly, for all $\lambda \in \Lambda$, the set $\bigcup_{\mu < \lambda} \tilde{G}_N^\mu \tilde{F}_M^\mu$ is a K -vector space basis of $\text{Hom}_{\mathcal{C}}(M, N)^{< \lambda}$.
- (iv) The sets $\tilde{G}_N^\mu \tilde{F}_M^\mu$ are pairwise disjoint for the various $\mu \in \Lambda$.

Proof.

Regarding (i) and (ii): We will show statements (i) and (ii) simultaneously by induction on the length l of a costandard filtration of N .

Base case: If $l = 1$ holds, then there exists an element $\lambda \in \Lambda$ such that $N \cong \nabla(\lambda)$.

We set

$$G_N^\mu := \begin{cases} \{c^\lambda\} & \text{if } \lambda = \mu \\ \emptyset & \text{else} \end{cases}$$

Here, the morphism c^λ is defined as in Section 2. If $\mu = \lambda$ holds, then $G_N^\mu = \{c^\lambda\}$ is a basis of $\text{Hom}_{\mathcal{C}}(\Delta(\mu), \nabla(\lambda)) \cong K$ since c^λ is non-zero. If on the other hand $\mu \neq \lambda$ holds, then $\text{Hom}_{\mathcal{C}}(\Delta(\mu), \nabla(\lambda)) \cong \text{Ext}_{\mathcal{C}}^0(\Delta(\mu), \nabla(\lambda)) = 0$ by Definition 1.1.3. Thus, G_N^μ is a basis of $\text{Hom}_{\mathcal{C}}(\Delta(\mu), N)$ for all $\mu \in \Lambda$. Next, define

$$\tilde{G}_N^\mu := \begin{cases} \{\pi^\lambda\} & \text{if } \lambda = \mu \\ \emptyset & \text{else} \end{cases}$$

By definition of c^λ (c.f. Section 2), the morphism π^λ extends c^λ along ι^λ .

Now, observe that the following chain of equalities holds

$$\tilde{G}_N \tilde{F}_M = \tilde{G}_N^\lambda \tilde{F}_M^\lambda = F_M^\lambda. \quad (11)$$

Thus, the set $\tilde{G}_N \tilde{F}_M = F_M^\lambda$ defines a basis of the K -vector space $\text{Hom}_{\mathcal{C}}(M, N) = \text{Hom}_{\mathcal{C}}(M, \nabla(\lambda))$.

To show statement (ii) in the case $l = 1$, pick $\mu \in \Lambda$ and let $\phi \in \text{span}_K(\tilde{G}_N^\mu \tilde{F}_M^\mu)$ be a non-zero morphism. Since $\text{span}_K(\tilde{G}_N^\mu \tilde{F}_M^\mu) = 0$ for $\mu \neq \lambda$ by definition of \tilde{G}_N^μ and by the convention mentioned in step (5) of Construction 3.1, we conclude $\mu = \lambda$. Thus, we have with equation 11

$$\phi \in \text{span}_K(F_M^\lambda) = \text{Hom}_{\mathcal{C}}(M, \nabla(\lambda)).$$

In particular, the image $\text{im}(\phi)$ is a non-zero subobject of $\nabla(\lambda)$. Hence, the socle $L(\lambda) \cong \text{Soc } \nabla(\lambda)$ is a subobject of $\text{im}(\phi)$. Thus, by [P70, Proposition 4.5.1] there exists a composition series

$$0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = \text{im}(\phi)$$

in which $L(\lambda)$ appears as an element. Since $L(\lambda)$ is simple, we know $A_1 = L(\lambda)$ and thus $0 \neq [\text{im}(\phi) : L(\lambda)] \stackrel{\text{def}}{=} \phi_\lambda$. This shows statement (ii) in the case that $l = 1$.

Induction step: For the induction step let $l \in \mathbb{N} \setminus \{1\}$. Let $N \in \mathcal{C}$ be an object with costandard filtration

$$0 = N_0 \subsetneq \cdots \subsetneq N_{l-1} \subsetneq N_l = N. \quad (12)$$

By the induction hypothesis, there exists a basis $G_{N_{l-1}}$ of $\text{Hom}_{\mathcal{C}}(\Delta(\lambda), N_{l-1})$ and a set of extensions $\tilde{G}_{N_{l-1}}$ such that the set $\tilde{G}_{N_{l-1}} \tilde{F}_M$ is a basis of $\text{Hom}_{\mathcal{C}}(M, N_{l-1})$ satisfying (ii). From said data we now construct a basis of $\text{Hom}_{\mathcal{C}}(M, N)$.

To do so, find $\lambda \in \Lambda$ with $N_l/N_{l-1} \cong \nabla(\lambda)$. Consider the short exact sequence

$$0 \rightarrow N_{l-1} \xrightarrow{\text{inc}} N \xrightarrow{\text{proj}} \nabla(\lambda) \rightarrow 0. \quad (13)$$

Let $\mu \in \Lambda$. Applying the left exact functor $\text{Hom}_{\mathcal{C}}(\Delta(\mu), -)$ to sequence 13 and using Corollary 1.4 yields the following short exact sequence in the category of finite-dimensional K -vector spaces:

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(\Delta(\mu), N_{l-1}) \xrightarrow{\text{inc}*} \text{Hom}_{\mathcal{C}}(\Delta(\mu), N) \xrightarrow{\text{proj}*} \text{Hom}_{\mathcal{C}}(\Delta(\mu), \nabla(\lambda)) \rightarrow 0. \quad (14)$$

Analogously, by applying the left exact functor $\text{Hom}_{\mathcal{C}}(M, -)$ to sequence 13 we obtain the following short exact sequence in the category of finite-dimensional K -vector spaces:

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(M, N_{l-1}) \xrightarrow{\text{inc}*} \text{Hom}_{\mathcal{C}}(M, N) \xrightarrow{\text{proj}*} \text{Hom}_{\mathcal{C}}(M, \nabla(\lambda)) \rightarrow 0. \quad (15)$$

Next, we define the set G_N^μ depending on $\mu \in \Lambda$.

In the case that $\mu \neq \lambda$, set

$$G_N^\mu := \text{inc}_*(G_{N_{l-1}}^\mu) \quad (16)$$

$$\tilde{G}_N^\mu := \text{inc}_*(\tilde{G}_{N_{l-1}}^\mu). \quad (17)$$

Since by Definition 1.1.3 we have $\text{Hom}_{\mathcal{C}}(\Delta(\mu), \nabla(\lambda)) = 0$ if $\mu \neq \lambda$, the morphism inc_* in sequence 14 is a K -linear isomorphism. Thus, the set G_N^μ is a basis of $\text{Hom}_{\mathcal{C}}(\Delta(\mu), N)$. Moreover, by construction the set \tilde{G}_N^μ is a set of extensions of G_N^μ along ι^μ .

In the case that $\mu = \lambda$, choose a morphism $g^\lambda: \Delta(\lambda) \rightarrow N$ such that $\text{proj} \circ g^\lambda = c^\lambda$. Here, the morphism c^λ is defined as in Section 2. Such a g^λ exists, since the sequence 14 is exact. Then the set

$$G_N^\mu := \text{inc}_*(G_{N_{l-1}}^\lambda) \cup \{g^\lambda\}$$

defines a basis of $\text{Hom}_{\mathcal{C}}(\Delta(\lambda), N)$, since $\{c^\lambda\}$ is a basis of $\text{Hom}_{\mathcal{C}}(\Delta(\lambda), \nabla(\lambda)) \cong K$ and any short exact sequence of vector spaces (in particular sequence 14) splits. Next, choose by the lifting lemma 2.1 any extension $\tilde{g}^\lambda: T(\lambda) \rightarrow N$ of g^λ along ι^λ and set

$$\tilde{G}_N^\lambda := \text{inc}_*(\tilde{G}_{N_{l-1}}^\lambda) \cup \{\tilde{g}^\lambda\}. \quad (18)$$

By construction the set \tilde{G}_N^λ is a set of extensions of G_N^λ along ι^λ . We omit the proof that with definitions 17 and 18 the set $\tilde{G}_N \tilde{F}_M = \bigcup_{\lambda \in \Lambda} \tilde{G}_N^\lambda \tilde{F}_M^\lambda$ is a basis of $\text{Hom}_{\mathcal{C}}(M, N)$. A proof is given in [AST18, Proposition 3.3.4]. That this basis satisfies (ii) is shown in detail in [BT22, p. 18].

To prove statements (iii) and (iv) choose a basis G_N and a set of extensions \tilde{G}_N satisfying (i) and (ii).

Regarding (iii): Let $\lambda \in \Lambda$. By assumption (ii), it suffices to prove that $\bigcup_{\mu \leq \lambda} \tilde{G}_N^\mu \tilde{F}_M^\mu$ spans $\text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$. We first convince ourselves that if $\mu \leq \lambda$ holds, then we have the inclusion

$$\tilde{G}_N^\mu \tilde{F}_M^\mu \subseteq \text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}. \quad (19)$$

To see this, let $\tilde{g}_i^\mu \circ \tilde{f}_j^\mu \in \tilde{G}_N^\mu \tilde{F}_M^\mu$. Set $c_{ij}^\mu := \tilde{g}_i^\mu \circ \tilde{f}_j^\mu$. If $(c_{ij}^\mu)_\nu \stackrel{\text{def}}{=} [\text{im}(c_{ij}^\mu): L(\nu)] \neq 0$ for $\nu \in \Lambda$, then $[\text{im}(\tilde{g}_i^\mu): L(\nu)] \neq 0$. This follows from Corollary 4.5 since by [P70, Lemma 4.3.6 a)] we have $\text{im}(\tilde{g}_i^\mu \circ \tilde{f}_j^\mu) \subseteq \text{im}(\tilde{g}_i^\mu)$. Thus, again by Corollary 4.5:

$$[T(\mu): L(\nu)] \geq [T(\mu)/\ker(\tilde{g}_i^\mu): L(\nu)] = [\text{im}(\tilde{g}_i^\mu): L(\nu)] > 0. \quad (20)$$

Since $T(\mu)$ is of highest weight μ by Definition 1.1.4, we conclude $\mu \geq \nu$. With the transitivity of the partial order, we hence have $\lambda \geq \nu$. This shows inclusion 19. From inclusion 19 we conclude

$$\text{span}_K \left(\bigcup_{\mu \leq \lambda} \tilde{G}_N^\mu \tilde{F}_M^\mu \right) \subseteq \text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}. \quad (21)$$

Now, to show that the two sets appearing in 21 are actually equal, pick a non-zero morphism $\phi \in \text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$. By assumption (i), we can write

$$\phi = \sum_{\mu \in \Lambda} a_{ij}^\mu c_{ij}^\mu$$

with $a_{ij}^\mu \in K$ and $c_{ij}^\mu \in \tilde{G}_N^\mu \tilde{F}_M^\mu$ for $\mu \in \Lambda$. Pick a maximal (not necessarily unique) $\mu \in \Lambda$ such that $a_{ij}^\mu \neq 0$ for some i and j . Define

$$\phi^\mu := \sum_{i,j} a_{ij}^\mu c_{ij}^\mu \quad \text{and} \quad \phi^{\neq \mu} := \sum_{\substack{\nu \neq \mu \\ i,j}} a_{ij}^\nu c_{ij}^\nu.$$

Since the set of c_{ij}^μ 's is K -linearly independent, we know that $\phi^\mu \in \text{span}_K(\tilde{G}_N^\mu \tilde{F}_M^\mu)$ is non-zero. With (ii) we conclude that the μ -multiplicity $(\phi^\mu)_\mu$ is non-zero. By the maximality of μ we have $(\phi^{\neq \mu})_\mu = 0$. Now, Lemma 4.8 yields the chain of equalities $\phi_\mu = (\phi^\mu + \phi^{\neq \mu})_\mu = (\phi^\mu)_\mu \neq 0$. By definition of $\text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$, we conclude $\mu \leq \lambda$. Since this inequality holds for any maximal μ , this implies $\phi \in \text{span}_K(\bigcup_{\mu \leq \lambda} \tilde{G}_N^\mu \tilde{F}_M^\mu)$.

Regarding (iv): Let $\phi \in \tilde{G}_N^\lambda \tilde{F}_M^\lambda \cap \tilde{G}_N^\mu \tilde{F}_M^\mu$. By definition of a basis we have $\phi \neq 0$. With property (ii) we conclude $\phi_\mu \neq 0$. Additionally, by property (iii) we know that $\phi \in \text{Hom}_{\mathcal{C}}(M, N)^{\leq \lambda}$. Thus, $\mu \leq \lambda$. By exchanging μ and λ one shows $\lambda \leq \mu$. With the antisymmetry of a partial order, we conclude $\lambda = \mu$. □

Remark 4.10.

One may exchange the roles of F_M and G_N in the basis theorem 4.9.

With Lemma 3.2.4. from Kian's talk, we obtain the following result.

Corollary 4.11. (Counting dimension via multiplicities in (co)standard filtrations)

We have

$$\dim_K(\text{Hom}_{\mathcal{C}}(M, N)) = \sum_{\lambda \in \Lambda} (M : \nabla(\lambda)) \cdot (N : \Delta(\lambda)).$$

The dependent basis theorem now allows us to prove the original basis theorem by a change-of-basis argument.

Proof of Theorem 4.1.

A detailed proof is given in [BT22, Proof of Theorem 3.2.]. The proof amounts to showing that a certain linear transformation is an isomorphism. □

The proof of the basis theorem 4.1 yields the subsequent proposition.

Proposition 4.12.

Any choice for $F_M^\lambda, G_N^\lambda, \tilde{F}_M^\lambda, \tilde{G}_N^\lambda$ in Construction 3.1 satisfies property (ii) of the dependent basis theorem 4.9.

Proposition 4.12 together with the dependent basis theorem 4.9 implies the following result.

Corollary 4.13.

Any choice for $F_M^\lambda, G_N^\lambda, \tilde{F}_M^\lambda, \tilde{G}_N^\lambda$ in Construction 3.1 satisfies properties (iii) and (iv) of the dependent basis theorem 4.9.

5 Combinatorial bases

We fix a tilting object $T \in \mathcal{C}$. The basis theorem 4.1 immediately gives us a way to construct bases of the endomorphism algebra $\text{End}_{\mathcal{C}}(T)$ of T . In this section, we equip these bases with extra (combinatorial) structure.

5.1 Standard bases

It turns out that the bases of the endomorphism algebra $\text{End}_{\mathcal{C}}(T)$ constructed in the basis theorem 4.1 already come with a canonical combinatorial structure, making them so-called *standard bases*.

Definition 5.1 (Du-Rui)

Let E be a finite-dimensional K -algebra. A standard basis of E consists of:

- (i) A K -basis \mathbf{B} of E ,
- (ii) A poset Λ and a function $f: \mathbf{B} \rightarrow \Lambda$. We write $\mathbf{B}^\lambda := f^{-1}(\lambda)$,
- (iii) For any $\lambda \in \Lambda$, indexing sets \mathbf{I}^λ and \mathbf{J}^λ such that:
 - (a) We can write $\mathbf{B}^\lambda = \{c_{ij}^\lambda \mid i \in \mathbf{I}^\lambda \text{ and } j \in \mathbf{J}^\lambda\}$,
 - (b) For any $\phi \in E$ and any $c_{ij}^\lambda \in \mathbf{B}^\lambda$, we have:

$$\phi \cdot c_{ij}^\lambda \equiv \sum_{k \in \mathbf{I}^\lambda} r_k^\lambda(\phi, i) c_{kj}^\lambda \pmod{E^{<\lambda}}, \quad (22)$$

$$c_{ij}^\lambda \cdot \phi \equiv \sum_{l \in \mathbf{J}^\lambda} r_l^\lambda(j, \phi) c_{il}^\lambda \pmod{E^{<\lambda}}, \quad (23)$$

where $r_k^\lambda(\phi, i)$, $r_l^\lambda(j, \phi) \in K$ are independent of j and i , respectively.

Here, we let $E^{<\lambda} := \text{span}_K(\bigcup_{\mu < \lambda} \mathbf{B}^\mu) \subseteq E$.

Theorem 5.2.

Any K -basis $\tilde{G}_T \tilde{F}_T$ of $\text{End}_{\mathcal{C}}(T)$ arising from Construction 3.1 carries a canonical structure of a standard basis.

Proof.

Firstly, note that by Corollary 4.13 the set \mathbf{B} is fibered over the poset Λ , i.e.

$$\tilde{G}_T \tilde{F}_T = \coprod_{\lambda \in \Lambda} \tilde{G}_T^\lambda \tilde{F}_T^\lambda. \quad (24)$$

For any $\lambda \in \Lambda$, we can thus set

$$\mathbf{B}^\lambda := \tilde{G}_T^\lambda \tilde{F}_T^\lambda. \quad (25)$$

Secondly, for any $\lambda \in \Lambda$, define

$$\mathbf{I}^\lambda := \{n \in \mathbb{N} \mid n \leq (T: \Delta(\lambda))\} \quad (26)$$

$$\mathbf{J}^\lambda := \{n \in \mathbb{N} \mid n \leq (T: \nabla(\lambda))\}. \quad (27)$$

By Lemma 3.2.4. from Kian's talk, we can index

$$\tilde{G}_T^\lambda = \{\tilde{g}_i^\lambda \mid i \in I^\lambda\} \quad (28)$$

$$\tilde{F}_T^\lambda = \{\tilde{f}_j^\lambda \mid j \in J^\lambda\}. \quad (29)$$

We can thus write

$$B^\lambda = \{\tilde{g}_i^\lambda \circ \tilde{f}_j^\lambda \mid i \in I^\lambda \text{ and } j \in J^\lambda\}. \quad (30)$$

Next, we prove equation 22. Let $\phi \in \text{End}_{\mathcal{C}}(T)$. Choose $\tilde{g}_i^\lambda \circ \tilde{f}_j^\lambda \in \tilde{G}_T^\lambda \tilde{F}_T^\lambda$. Set $c_{ij}^\lambda := \tilde{g}_i^\lambda \circ \tilde{f}_j^\lambda$. Since G_T^λ is a basis of $\text{Hom}(\Delta(\lambda), T)$, we find $r_k^\lambda(\phi, i) \in K$ such that

$$\phi \circ g_i^\lambda = \sum_{k \in I^\lambda} r_k^\lambda(\phi, i) g_k^\lambda. \quad (31)$$

Since for each $k \in I^\lambda$ the morphism \tilde{g}_k^λ extends g_k^λ along ι^λ , equation 31 yields

$$\phi \circ \tilde{g}_i^\lambda \circ \iota^\lambda = \sum_{k \in I^\lambda} r_k^\lambda(\phi, i) (\tilde{g}_k^\lambda \circ \iota^\lambda). \quad (32)$$

Since composition in an abelian category is bilinear, this amounts to

$$(\phi \circ \tilde{g}_i^\lambda - \sum_{k \in I^\lambda} r_k^\lambda(\phi, i) \tilde{g}_k^\lambda) \circ \iota^\lambda = 0. \quad (33)$$

Note that $\text{Hom}_{\mathcal{C}}(T(\lambda), T) = \text{Hom}_{\mathcal{C}}(T(\lambda), T)^{\leq \lambda}$. The inclusion \supseteq is clear. For the other inclusion, let $\psi: T(\lambda) \rightarrow T$ with $\psi_\mu \neq 0$. Then

$$[T(\lambda): L(\mu)] \geq [T(\lambda)/\ker(\psi): L(\mu)] = [\text{im}(\psi): L(\mu)] > 0.$$

Since by assumption $T(\lambda)$ is of highest weight λ , we conclude $\mu \leq \lambda$. The inclusion \subseteq follows.

In particular, we see that $\psi := \phi \circ \tilde{g}_i^\lambda - \sum_{k \in I^\lambda} r_k^\lambda(\phi, i) \tilde{g}_k^\lambda \in \text{Hom}_{\mathcal{C}}(T(\lambda), T)^{\leq \lambda}$. We even know that $\psi \in \text{Hom}_{\mathcal{C}}(T(\lambda), T)^{< \lambda}$. To see this, note that by the universal property of the kernel and by 33 there exists a monomorphism $\Delta(\lambda) \hookrightarrow \ker(\lambda)$. Thus, by Corollary 4.5 we have

$$[\ker(\psi): L(\lambda)] \geq [\Delta(\lambda): L(\lambda)]. \quad (34)$$

Since both $T(\lambda)$ and $\Delta(\lambda)$ are of highest weight λ , we conclude with one of the isomorphism theorems:

$$\begin{aligned} 0 = [T(\lambda): L(\lambda)] - [\Delta(\lambda): L(\lambda)] &\stackrel{34}{\geq} [T(\lambda): L(\lambda)] - [\ker(\psi): L(\lambda)] \\ &\stackrel{4.4}{\geq} [T(\lambda)/\ker(\psi): L(\lambda)] \\ &= [\text{im}(\psi): L(\lambda)]. \end{aligned}$$

Next, note that $\text{im}(\psi \circ \tilde{f}_j^\lambda) \subseteq \text{im}(\psi)$ by [P70, Lemma 4.3.6 a)]. Thus, by Proposition 4.4 we know

$$\phi \circ \tilde{g}_i^\lambda \circ \tilde{f}_j^\lambda - \sum_{k \in I^\lambda} r_k^\lambda(\phi, i) \tilde{g}_k^\lambda \circ \tilde{f}_j^\lambda = \psi \circ \tilde{f}_j^\lambda \in \text{Hom}_{\mathcal{C}}(T, T)^{< \lambda}. \quad (35)$$

Since by Corollary 4.13 the following two sets

$$\text{Hom}_{\mathcal{C}}(T, T)^{<\lambda} = \text{span}_K(\bigcup_{\mu < \lambda} \tilde{G}_T^\lambda \tilde{F}_T^\lambda) \quad (36)$$

are equal, we therefore have shown

$$\phi \circ c_{ij}^\lambda \equiv \sum_{k \in I^\lambda} r_k^\lambda(\phi, i) c_{kj}^\lambda \pmod{E^{<\lambda}}. \quad (37)$$

Equation 23 is proved analogously. \square

5.2 Cellular bases

We recall the definition of a cellular basis as presented in Muskan's talk.

Definition 5.3 (Graham-Lehrer)

Let E be a finite-dimensional K -algebra. A cellular basis of E consists of a standard basis B together with an involutive antialgebra morphism $*: E \rightarrow E$ such that the following two equalities hold:

- (i) $I^\lambda = J^\lambda$ for every $\lambda \in \Lambda$,
- (ii) $c_{ij}^* = c_{ji}$ for every $(i, j) \in I^\lambda \times J^\lambda$.

We will now turn some of the bases constructed in the basis theorem 4.1 into cellular bases. To do so, we need the following definitions.

Definition 5.4

A strong duality on \mathcal{C} consists of:

- (i) A contravariant K -linear endofunctor $D: \mathcal{C} \rightarrow \mathcal{C}$ on \mathcal{C} ,
- (ii) A natural isomorphism of K -linear functors

$$\xi: \text{id}_{\mathcal{C}} \rightarrow D^2$$

that satisfies

$$\text{id}_{D(X)} = D(\xi_X) \circ \xi_{D(X)}$$

for all objects $X \in \mathcal{C}$. As a commutative diagram:

$$\begin{array}{ccc} D(X) & \xrightarrow{\text{id}_{D(X)}} & D(X) \\ \xi_{D(X)} \searrow & & \nearrow D(\xi_X) \\ & D^3(X) & \end{array}$$

Definition 5.5

Let $D: \mathcal{C} \rightarrow \mathcal{C}$ be a strong duality on \mathcal{C} . A self-dual object (X, ϕ) in \mathcal{C} consists of an object $X \in \mathcal{C}$ and an isomorphism $\phi: D(X) \xrightarrow{\sim} X$.

Construction 5.6.

Let (D, ξ) be a strong duality on \mathcal{C} . Let (X, ϕ) be a self-dual object in \mathcal{C} . Define the K -antialgebra morphism

$$\alpha_\phi^{-1} \in \text{End}_{\mathcal{C}}(X)$$

by letting $\alpha_\phi^{-1}(\varphi) := \phi \circ D(\varphi) \circ \phi^{-1}$ for all $\varphi \in \text{End}_{\mathcal{C}}(X)$. Define

$$a := \phi \circ D(\phi^{-1}) \circ \xi_X \in \text{End}_{\mathcal{C}}(X).$$

Using the naturality of ξ , one sees that

$$\alpha_\phi^{-2}(\varphi) = a \circ \varphi \circ a^{-1}$$

for all $\varphi \in \text{End}_{\mathcal{C}}(X)$. In particular, α_ϕ^{-1} has a two-sided inverse

$$\alpha_\phi \in \text{End}(X).$$

Definition 5.7

Let (D, ξ) be a strong duality on \mathcal{C} . A self-dual object (X, ϕ) is called a fixed point of D if α_ϕ from Construction 5.6 is an anti-involution, i.e. if

$$a \stackrel{\text{def}}{=} \phi \circ D(\phi^{-1}) \circ \xi_X = \text{id}_X.$$

Definition 5.8

An object $X \in \mathcal{C}$ is called fixed by D if there exists an isomorphism $\phi: D(X) \xrightarrow{\sim} X$ such that the pair (X, ϕ) is a fixed point of D .

A proof of the following lemma is given in [BT22, Lemma 3.13].

Lemma 5.9.

Let D be a strong duality on \mathcal{C} that exchanges standard and costandard objects, i.e.

$$D(\nabla(\lambda)) \cong \Delta(\lambda)$$

for all $\lambda \in \Lambda$. Then we have:

- (i) Every simple object $L(\lambda)$ is fixed by D ,
- (ii) Every tilting object $T \in \mathcal{C}$ admits a self-dual structure, i.e. there exists an isomorphism $\phi: D(T) \xrightarrow{\sim} T$.

Definition 5.10

A standard duality on \mathcal{C} is a strong duality D on \mathcal{C} such that:

(i) The duality D exchanges standard and costandard objects, i.e. for all $\lambda \in \Lambda$ we have

$$D(\nabla(\lambda)) \cong \Delta(\lambda),$$

(ii) Every indecomposable tilting object $T(\lambda)$ is a fixed point, i.e. for every $\lambda \in \Lambda$ and every isomorphism $\phi: D(T(\lambda)) \xrightarrow{\sim} T(\lambda)$, the pair $(T(\lambda), \phi)$ is a fixed point of D .

Remark 5.11.

Lemma 5.9 ensures that condition (ii) in Definition 5.10 is not void (, i.e. that there exists an isomorphism $\phi: D(T(\lambda)) \xrightarrow{\sim} T(\lambda)$).

Construction 5.12.

Let D be a standard duality on \mathcal{C} . Let $T \in \mathcal{C}$ be a tilting object. We will sketch how some of the bases of $\text{End}(T)$ constructed in the basis theorem carry a cellular structure.

For this purpose, choose for every $\lambda \in \Lambda$ a basis G_T^λ of $\text{Hom}_{\mathcal{C}}(\Delta(\lambda), T)$. For any $\lambda \in \Lambda$, then pick a set of extensions \tilde{G}_T^λ of G_T^λ along the monomorphism ι^λ as in Construction 3.1.

Since the full subcategory \mathcal{C}^t consisting of tilting objects in \mathcal{C} is a Krull-Schmidt category (as shown in Corollary 3.2.6.3. of Kian's talk), the tilting object T is a finite direct sum of indecomposable tilting objects $T(\lambda)$. By property (ii) of a standard duality and since the functor D is required to be K -linear, we therefore know that the tilting object T is fixed by D . In other words, we can choose an isomorphism

$$\phi_T: D(T) \xrightarrow{\sim} T$$

such that

$$\phi_T \circ D(\phi_T^{-1}) \circ \xi_T = \text{id}_T.$$

Using this isomorphism, one constructs for every $\lambda \in \Lambda$ a basis $D(G_T^\lambda)$ of $\text{Hom}_{\mathcal{C}}(T, \nabla(\lambda))$ and a corresponding set of lifts $D(\tilde{G}_T^\lambda)$. For details see [BT22, p. 24]. We then let

$$D(\tilde{G}_T) := \coprod_{\lambda \in \Lambda} D(\tilde{G}_T^\lambda).$$

Theorem 5.13.

Let D be a standard duality on \mathcal{C} . Let $T \in \mathcal{C}$ be a tilting object. Choose a basis G_T and a set of extensions \tilde{G}_T . Then the standard basis $\tilde{G}_T D(\tilde{G}_T)$ of $\text{End}_{\mathcal{C}}(T)$ together with the involutive antialgebra morphism α_{ϕ_T} (as defined in Construction 5.6) define a cellular basis.

Proof.

Note that the duality D is fully faithful and exchanges standard and costandard objects. Together with the fact that the tilting object T admits a self-duality, this implies for all $\lambda \in \Lambda$:

$$\text{Hom}_{\mathcal{C}}(\Delta(\lambda), T) \cong \text{Hom}_{\mathcal{C}}(D(T), D(\Delta(\lambda))) \cong \text{Hom}_{\mathcal{C}}(T, \nabla(\lambda)). \quad (38)$$

With Lemma 3.2.4. from Kian's talk, we conclude $\mathbf{I}^\lambda = \mathbf{J}^\lambda$.

A detailed proof of the fact that the equality $\alpha_{\phi_T}(c_{ij}) = c_{ji}$ holds for all $(i, j) \in \mathbf{I}^\lambda \times \mathbf{J}^\lambda$ is given in [BT22, Theorem 3.16]. \square

References

- [AST18] H. H. Andersen, C. Stroppel and D. Tubbenhauer, *Cellular structures using U_q -tilting modules*, Pacific Journal of Mathematics, vol. 292, 2018. MR3708257
- [BT22] G. Bellamy and U. Thiel, *Cellularity of endomorphism algebras of tilting objects*, Advances in Mathematics, vol. 404, 2022. MR4413216
- [P70] B. Pareigis, *Categories and functors*, Pure and Applied Mathematics, Vol. 39, Academic Press, New York-London, 1970. MR0265428