What is dense about the Jacobson density theorem?

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Let R be a ring. We recall the following standard formulation of the Jacobson density theorem, e.g. [La02, §17, Thm 3.2]:

Theorem 0.1. (Jacobson) Let M be a *semisimple* R-module. For every endomorphism

$$f \in \mathrm{End}_{\mathrm{End}_R(M)}(M)$$

and every finite set $F \subseteq M$, there exists an element $x \in R$ such that

$$f(m) = x \cdot m \quad \text{for all} \quad m \in F.$$

We explain the name density theorem, using elementary topology: Let M be a (not neccessarily semisimple) R-module. Consider the ring homomorphism

$$\varphi_{\mathrm{Jac}} \colon R \longrightarrow \mathrm{End}_{\mathrm{End}_R(M)}(M); \qquad x \mapsto \lambda_x,$$

where $\lambda_x \colon M \to M$ denotes left multiplication by x, that is,

$$\lambda_x(m) = x \cdot m \quad \text{for all } m \in M.$$

We endow the set M with the discrete topology. On

$$\mathrm{End}_{\mathrm{End}_R(M)}(M)$$

we consider the compact-open topology. Recall that this topology has as subbase

$$\{W(K, U) \mid K \subseteq M \text{ compact}, U \subseteq M \text{ open}\},$$

where

$$W(K, U) := \{g \in \mathrm{End}_{\mathrm{End}_R(M)}(M) \mid g(K) \subseteq U\}.$$

Proposition 0.2.
The image of the ring homomorphism $\varphi_{\mathrm{Jac}} \colon R \to \mathrm{End}_{\mathrm{End}_R(M)}(M)$ is dense in $\mathrm{End}_{\mathrm{End}_R(M)}(M)$ if and only if for every morphism $f \in \mathrm{End}_{\mathrm{End}_R(M)}(M)$ and any finite set $F \subseteq M$, there exists a ring element $x \in R$ such that $f(m) = x \cdot m$ for all $m \in F$.

Proof.
Let $f \in \mathrm{End}_{\mathrm{End}_R(M)}(M)$. Take a finite set $F \subseteq M$. By assumption, for any open neighbourhood $U \subseteq \mathrm{End}_{\mathrm{End}_R(M)}(M)$ of f, we have $U \cap \mathrm{Im}(\varphi_{\mathrm{Jac}}) \neq \varnothing$. Consider the set

$$U := \bigcap_{m \in F} W(\{m\}, \{f(m)\}).$$

This is an open neighbourhood of f. Thus, there exists an $x \in R$ such that $\varphi_{\mathrm{Jac}}(x) \in U$. By definition of U, for such an $x \in R$, we have $f(m) = \varphi_{\mathrm{Jac}}(x)m$ for all $m \in F$.

Conversely, assume that for every morphism $f \in \mathrm{End}_{\mathrm{End}_R(M)}(M)$ and any finite set $F \subseteq M$, there exists a ring element $x \in R$ such that $f(m) = \varphi_{\mathrm{Jac}}(x)m$ for all $m \in F$. Next, let $f \in \mathrm{End}_{\mathrm{End}_R(M)}(M)$. Let $U \subseteq \mathrm{End}_{\mathrm{End}_R(M)}(M)$ be an open neighbourhood of f. By definition of the compact-open topology, we can write U as

$$U = \bigcup_{i \in I} \Big(\bigcap_{j \in J_i} W(K_{ij}, V_{ij}) \Big),$$

where each indexing set J_i is finite, each $K_{ij} \subseteq M$ is compact, and each V_{ij} is a subset of M. Since the topology on M is discrete, each K_{ij} is finite. Now, let $k \in I$ such that $f \in \bigcap_{j \in J_k} W(K_{kj}, V_{kj})$. Define $K \coloneqq \bigcup_{j \in J_k} K_{kj}$. Note that, as a finite union of finite sets, K is a finite subset of M. By assumption, we can thus find $x \in R$ such that $f(m) = \varphi_{\mathrm{Jac}}(x)m$ for all $m \in K$. For such an $x \in R$, we therefore have $\varphi_{\mathrm{Jac}}(x) \in \bigcap_{j \in J_k} W(K_{kj}, V_{kj}) \subseteq U$. Consequently, the image of the ring homomorphism $\varphi_{\mathrm{Jac}} \colon R \to \mathrm{End}_{\mathrm{End}_R(M)}(M)$ is dense in $\mathrm{End}_{\mathrm{End}_R(M)}(M)$. □

As an immediate consequence, the Jacobson density theorem has the following reformulation:

For any semisimple R-module, the image of the ring homomorphism

$$\varphi_{Jac} \colon R \longrightarrow \mathrm{End}_{\mathrm{End}_R(M)}(M)$$

is dense in $\mathrm{End}_{\mathrm{End}_R(M)}(M)$*, where R carries the discrete topology and* $\mathrm{End}_{\mathrm{End}_R(M)}(M)$ *the compact-open topology.*

References

[La02] S. Lang, *Algebra*, Graduate Texts in Mathematics, vol. 211, 3rd ed, Springer-Verlag, New York, 2002.